

Non-singular spinors in gravity with propagating torsion

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(Dated: February 14, 2017)

We demonstrate that when the propagating torsion massive axial vector is coupled to spinor fields then gravitational singularities do not form necessarily.

I. INTRODUCTION

With aLIGO's recent detection of gravitational waves, we finally completed the list of original tests that Einstein gravity had to pass; with all predictions successfully confirmed and being built on properly defined mathematical foundations, Einstein gravitation is the only physical construction that could claim to be both phenomenologically very accurate and theoretically well established.

The only problem that Einstein gravitation might have is the circumstance for which matter should form singularities that would invalidate the theory at high energy.

We must stress two facts: one is that this circumstance is genuinely theoretical, not an observation; and the other is that despite being theoretical, it is nevertheless not a failure of the underlying mathematical structure.

But even if all is correct in its mathematical structure nor is there any mismatch with observations, there is still the theoretical possibility that Einstein gravitation would break down at high energies due to the Penrose-Hawking singularity theorems, stating that singularities are bound to form for matter satisfying certain energy conditions.

If we want Einstein gravity to be purged of this issue we can only try to avoid these energy conditions and thus finding matter distributions that do not verify them, but for the moment we know only one form matter that can form stable structures, and because such a matter is given by fermion fields which verify even the strongest of those energy conditions then singularities appear necessarily.

However, fermion fields are described by spinor fields, and because spinors have also a spin beside an energy it is conceptually compelling to consider a theory in which also the spin and not only the energy be coupled to some geometrical quantity: in the most general background we can have, that is the Riemann-Cartan geometry in which the metric structure is supplemented by the presence of the torsion tensor [1–4], the most exhaustive coupling can be achieved by having energy coupled to curvature supplemented by spin coupled to torsion in Einstein gravity's Sciama-Kibble completion [5, 6]. This ESK gravity is for this reason naturally equipped to describe space-times in which $\frac{1}{2}$ -spin Dirac fields can find their place [7].

In the ESK theory of the Dirac field, or DESK theory, it is easy to demonstrate that the presence of torsion can be turned into a non-linear potential for the spinor with the consequence that there arises a contact interaction of weak repulsive character; repulsive forces have a positive potential that increases the energy content in the

source of Einstein equations and therefore singularities form even more easily, as proven by Kerlick [8].

The DESK theory is more general but nevertheless it enhances the energy condition and thus the singularity.

The reason for this is that the DESK theory of gravity, albeit a generalization, is the simplest of generalizations, where torsion is coupled to spin and energy is coupled to curvature in terms of the same coupling constant, that is the Newton constant: this is too restrictive as there is no way of justifying why there ought be the same coupling for different field equations involving independent fields.

In the most general situation, where the independence of torsion and metric gets reflected into the independence of their field equations and therefore in the independence of their coupling constants [9], the torsion-spin coupling constant is in general not the Newton constant, and thus it is allowed to be larger and with different sign; therefore the torsionally-induced spin-spin force is permitted to be stronger and attractive [10]: the Kerlick's generalization of Penrose-Hawking singularity theorems may revert.

This may look good enough, but once again there are two problems: first, albeit singularities might be avoided, it is better if they are not even possible, second, although the theory presented in [10] is the most general in terms of the independence of the degrees of freedom, nevertheless the torsional degrees of freedom do not propagate.

So the torsion field would fail to be consistent.

A consistent theory in which all fields propagate, with field equations being defined at the least-order derivative in the most general case, is in [11], where also the causal structure of the field equations is studied: in this theory, the hope is that the torsionally-induced spin-spin force is attractive, so that singularities become impossible.

In this paper, we will study this circumstance.

II. PROPAGATING TORSION IN GRAVITY FOR SPINORS

In this paper we will refer to [11] for all notations; and in particular, we recall that the torsion may be assumed to be completely antisymmetric with no loss of generality.

In the space-time the completely antisymmetric torsion is equivalent to the dual of an axial vector W^μ and so that the torsion tensor turns out to be equivalent to an axial-vector massive field verifying Proca field equations and with the corresponding energy as a source of the Einstein gravitational field equations; defining $(\partial W)_{\alpha\nu}$ as curl of

the axial-vector torsion, $R_{\mu\nu}$ as Ricci curvature, together with γ^μ as Clifford matrices such that $\pi = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the parity-odd gamma matrix and $\bar{\psi}$ and ψ as the Dirac conjugate spinors such that ∇_μ is the spinorial covariant derivative, then we have that the expression given by

$$\mathcal{L} = \frac{1}{4}(\partial W)^2 - \frac{1}{2}M^2W^2 + \frac{1}{k}R - i\bar{\psi}\gamma^\mu\nabla_\mu\psi + X\bar{\psi}\gamma^\mu\pi\psi W_\mu + m\bar{\psi}\psi \quad (1)$$

is the most general least-order derivative Lagrangian that is possible: k is (related to) the Newton gravitational constant while X is the torsion-spin coupling constant as well as M and m being torsion and spinor masses.

The torsion axial-vector field equations are given by

$$\nabla_\rho(\partial W)^{\rho\mu} + M^2W^\mu = X\bar{\psi}\gamma^\mu\pi\psi \quad (2)$$

having the structure of Proca equations while the gravitational symmetric tensor field equations are given by

$$R^{\rho\sigma} - \frac{1}{2}Rg^{\rho\sigma} = \frac{k}{2}[\frac{1}{4}(\partial W)^2g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^\rho{}_\alpha + M^2(W^\rho W^\sigma - \frac{1}{2}W^2g^{\rho\sigma}) + \frac{i}{4}(\bar{\psi}\gamma^\rho\nabla^\sigma\psi - \nabla^\sigma\bar{\psi}\gamma^\rho\psi + \bar{\psi}\gamma^\sigma\nabla^\rho\psi - \nabla^\rho\bar{\psi}\gamma^\sigma\psi) - \frac{1}{2}X(W^\sigma\bar{\psi}\gamma^\rho\pi\psi + W^\rho\bar{\psi}\gamma^\sigma\pi\psi)] \quad (3)$$

having the structure of Einstein field equations, and finally the spinorial field equations given according to

$$i\gamma^\mu\nabla_\mu\psi - XW_\sigma\gamma^\sigma\pi\psi - m\psi = 0 \quad (4)$$

as Dirac equations; by taking the divergence of (2) and by contracting (3), and employing (4), gives

$$M^2\nabla_\mu W^\mu = 2Xmi\bar{\psi}\pi\psi \quad (5)$$

and

$$-R = \frac{k}{2}(-M^2W^2 + m\Phi) \quad (6)$$

the former being the partially conserved axial-vector for torsion: the latter substituted back into (3) gives us

$$R^{\rho\sigma} = \frac{k}{2}[\frac{1}{4}(\partial W)^2g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^\rho{}_\alpha + \frac{i}{4}(\bar{\psi}\gamma^\rho\nabla^\sigma\psi - \nabla^\sigma\bar{\psi}\gamma^\rho\psi + \bar{\psi}\gamma^\sigma\nabla^\rho\psi - \nabla^\rho\bar{\psi}\gamma^\sigma\psi) + M^2W^\rho W^\sigma - \frac{1}{2}m\bar{\psi}\psi g^{\rho\sigma} - \frac{1}{2}X(W^\sigma\bar{\psi}\gamma^\rho\pi\psi + W^\rho\bar{\psi}\gamma^\sigma\pi\psi)] \quad (7)$$

which are equivalent to the original ones but best suited in view of their application to the singularity theorems.

In fact, for the singularity theorems in Einstein gravity the strongest energy condition must read

$$R^{\rho\sigma}u_\rho u_\sigma \geq 0 \quad (8)$$

where u^α are time-like vectors: in our case we have that

$$[\frac{1}{4}(\partial W)^2g^{\rho\sigma} - (\partial W)^{\sigma\alpha}(\partial W)^\rho{}_\alpha + \frac{i}{2}(\bar{\psi}\gamma^\rho\nabla^\sigma\psi - \nabla^\sigma\bar{\psi}\gamma^\rho\psi) + M^2W^\rho W^\sigma - \frac{1}{2}m\bar{\psi}\psi g^{\rho\sigma} - XW^\sigma\bar{\psi}\gamma^\rho\pi\psi]u_\rho u_\sigma \geq 0 \quad (9)$$

as the energy condition that now has to be determined.

Because u^α is time-like, it is possible to find a frame in which it only has the time component, and in that frame condition (9) can be worked out with (4) to give that

$$\frac{1}{4}(\partial W)^2 - (\partial W)^{0k}(\partial W)^0{}_k + \frac{i}{2}(\bar{\nabla}\bar{\psi}\cdot\vec{\gamma}\psi - \bar{\psi}\vec{\gamma}\cdot\bar{\nabla}\psi) + M^2|W^0|^2 + \frac{1}{2}m\bar{\psi}\psi - X\vec{W}\cdot\bar{\psi}\vec{\gamma}\pi\psi \geq 0 \quad (10)$$

in which we clearly see the kinetic energies, mass terms and interactions of the torsion and the spinor field.

With this theory at our disposal we may now proceed in studying its consequences for the singularities.

III. SINGULARITY AVOIDANCE

To begin, we try to assess what happens in the effective approximation, when in (2) the torsion mass dominates and then we may approximate them down to

$$M^2W^\mu \approx X\bar{\psi}\gamma^\mu\pi\psi \quad (11)$$

which allows us to integrate torsional degrees of freedom away: the energy condition (10) thus becomes

$$\frac{i}{2}(\bar{\nabla}\bar{\psi}\cdot\vec{\gamma}\psi - \bar{\psi}\vec{\gamma}\cdot\bar{\nabla}\psi) - \frac{X^2}{M^2}|\bar{\psi}\pi\psi|^2 - \frac{X^2}{M^2}|\bar{\psi}\psi|^2 + \frac{1}{2}m\bar{\psi}\psi \geq 0 \quad (12)$$

where spinors have been re-arranged with Fierz identities and thus showing very clearly that the contribution of the non-linear potential are negative, and thus attractive.

In the case of non-relativistic limit, in which the kinetic energy and the Takabayashi angle are negligible, we get

$$|\bar{\psi}\psi| \leq \frac{mM^2}{2X^2} \quad (13)$$

and for field densities that are large enough as one would expect in the case of singularity formation this condition would be violated, and singularities are impossible.

However, in the case of high density it may well happen that effective approximations and non-relativistic limits cannot be valid, so we must consider (10) as is, and albeit masses are negligible, all remaining terms have a similar scaling behaviour, and a complete study can only be done by solving the system of field equations exactly.

IV. CONCLUSION

In this paper we have shown that for the most general least-order derivative Lagrangian of propagating torsion in gravity for spinors, the fact that the torsion propagates means that in the torsion field equations there must be a mass term, which is positive, and that within the Einstein field equations there must be a torsion energy contribution, which is positive, and these two facts together imply that torsionally-induced spin-spin forces be attractive, in the effective approximation; in turn this implies that the

material singularity formation under gravitation collapse are impossible, if the large mass approximations remain valid for large densities. If they do not, then singularity formation may or may not occur, but a proper treatment can be given only by solving the field equations exactly.

Whether the large mass approximations hold or not we cannot say because we do not know what the value of the torsion mass actually is: but either because singularities are impossible or because singularities might appear but not as a necessity, in any case we can say that the claims for which singularities cannot be avoided are void.

As a consequence, all the statements are based on the necessity of singularity formation to imply the inevitable

break-down of Einstein gravity are untenable.

When spinor fields are present Einstein gravity must be complemented by spin-torsion interactions and in the case in which the torsion propagates: the ensuing theory is not necessarily affected by singularities, and if the torsion mass is large enough to allow the limit given by the effective approximation then singularities are impossible.

At least in the effective approximation, the torsion-spin coupling is described by a potential that resembles the degeneracy pressure balancing the gravitational collapse thus ensuring the stability of neutron stars.

Could this torsion-spin interaction have anything to do with the Pauli exclusion principle?

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